I. Linear equations

Lesson Overview

- Write $y'(x)$ as $\frac{dy}{dx}$ and then try to separate the variables as follows:

$$(\text{function of } y)dy = (\text{function of } x)dx$$

- Then you can integrate both sides and solve for $y(x)$.

Notes

- Some equations are both linear and separable, so you can use either technique to solve them. But separation is usually easier.
- When you're integrating both sides of the equation, the C is very important. And it's important that you add it when you do the integration and keep track of it in the ensuing algebra.

Example I

Find the general solution to the following differential equation:

$$y'(x) = \frac{1}{2}y(x)$$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}y \\
\frac{dy}{y} &= \frac{1}{2}dx \\
\ln|y| &= \frac{1}{2}x + C && \left\{\begin{array}{l}\text{No } C \text{ needed on the } y \\ \text{side because it would be} \\ \text{absorbed into the other } C.\end{array}\right\} \\
|y| &= e^{\frac{1}{2}x+C} \\
&= ke^{\frac{1}{2}x} && \left\{\begin{array}{l}\text{[Omit discussion of positives} \\ \text{and negatives.]}\end{array}\right\} \\
\text{General solution:} \quad y &= \boxed{ke^{\frac{1}{2}x}}
\end{aligned}
$$

Use IC (if given) to find k.

Example II

Solve the following initial value problem:

$$yy' + x = 0, y(0) = 3$$

$$
\begin{aligned}
y' &= -\frac{x}{y} \\
\int y\,dy &= -\int x\,dx \\
\frac{y^2}{2} &= \frac{x^2}{2} + C \\
x^2 + y^2 &= k \\
y(0) = 3 &\implies k = 9 \\
&\implies \boxed{x^2 + y^2 = 9}
\end{aligned}
$$

Example III

Determine if the differential equation

$$y'(x) + xy = x^3$$

is separable.

$$\begin{aligned} \frac{dy}{dx} + xy &= x^3 \\ \frac{dy}{dx} &= x^3 - xy \end{aligned}$$

We can't factor the RHS into the form $f(x)g(y)$, so this equation is $\boxed{\text{not separable}}$. However, it is linear, since it has the form $y' + Py = Q$, so we could solve using the technique for linear equations.

Example IV

Solve the initial value problem:

$$3x - 6y\sqrt{x^2+1}\frac{dy}{dx} = 0, y(0) = 4$$

$$\begin{aligned} 6y\sqrt{x^2+1}\frac{dy}{dx} &= 3x \\ 2y\,dy &= \frac{x\,dx}{\sqrt{x^2+1}} \qquad \{\text{Let } u := x^2+1, du = 2x\,dx.\} \\ y^2 &= \sqrt{x^2+1} + C \\ y &= \sqrt{\sqrt{x^2+1} + C} \\ 4 &= \sqrt{1+C} \\ C &= 15 \\ y &= \boxed{\sqrt{\sqrt{x^2+1} + 15}} \end{aligned}$$

Example V

Solve the initial value problem:

$$y'(x) = x^2 y, y(0) = 7$$

$$\begin{aligned}
y'(x) &= x^2 y \\
\frac{dy}{dx} &= x^2 y \\
\frac{dy}{y} &= x^2 dx \\
\ln|y| &= \frac{x^3}{3} + C \\
y &= e^{\frac{x^3}{3}+C} \\
&= ke^{\frac{x^3}{3}} \\
y(0) = 7 &\implies \boxed{y = 7e^{\frac{x^3}{3}}}
\end{aligned}$$