XII. Series solutions

Lesson Overview

- Guess a power series solution and calculate its derivatives:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y'(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} \\ &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \end{aligned}$$

- Guess a power series solution and calculate its derivatives:

$$\begin{aligned} y''(x) &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \end{aligned}$$

- Plug your series into the differential equation.

- To combine the series:

  1. First match exponents on x by shifting indices.
  2. Then match starting indices by pulling out initial terms.

- Find a recurrence relation on the coefficients.
- Solve for higher coefficients in terms of lower ones.
- Use your coefficients to build your solutions.

Example I

Guess a series solution to the differential equation:

$$y' - 3x^2 y = 0$$

Plug in the solution and find a recurrence relation on the coefficients.

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} \\
3x^2 y &= \sum_{n=0}^{\infty} 3 a_n x^{n+2}
\end{aligned}$$

1. First match exponents on x by shifting indices using the mnemonic.
2. Then match starting indices by pulling out initial terms.

1. **Match exponents**:

$$\begin{aligned}
\sum_{n=0}^{\infty} 3 a_n x^{n+2} &= \sum_{n=2}^{\infty} 3 a_{n-2} x^n \\
\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n
\end{aligned}$$

2. **Match starting indices**:

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = a_1+2a_2x+\sum_{n=2}^{\infty}(n+1)a_{n+1}x^n$$

Plug in to the DE:

$$\begin{aligned} a_1 + 2a_2x + \sum_{n=2}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=2}^{\infty}3a_{n-2}x^n &= 0 \\ a_1 + 2a_2x + \sum_{n=\boxed{2}}^{\infty}\left[(n+1)a_{n+1} - 3a_{n-2}\right]x^n &= 0[+0x+0x^2+\cdots] \end{aligned}$$

Think of this as a big polynomial [add in terms on RHS]:

$$\begin{array}{lll} \underline{\text{const}}: & a_1 = 0 & \Longrightarrow a_1 = 0 \\ \underline{x}: & 2a_2 = 0 & \Longrightarrow a_2 = 0 \\ \underline{x^n}: & \boxed{(n+1)a_{n+1} - 3a_{n-2} = 0} & \text{for } n \geq \boxed{2}. \end{array}$$

Example II

Use the recurrence relation derived above to solve $y' - 3x^2y = 0$.

$$\begin{array}{lll} \underline{\text{const}}: & a_1 = 0 & \Longrightarrow a_1 = 0 \\ \underline{x}: & 2a_2 = 0 & \Longrightarrow a_2 = 0 \\ \underline{x^n}: & (n+1)a_{n+1} - 3a_{n-2} = 0 & \text{for } n \geq \boxed{2}. \end{array}$$

Example II

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$$\begin{array}{lll} \underline{\text{const}}: & a_1 = 0 & \Longrightarrow a_1 = 0 \\ \underline{x}: & 2a_2 = 0 & \Longrightarrow a_2 = 0 \\ \underline{x^n}: & (n+1)a_{n+1} - 3a_{n-2} = 0 & \text{for } n \geq \boxed{2}. \end{array}$$

$$a_{n+1} = \frac{3a_{n-2}}{n+1}$$

$n \geq 2$: $n = 2 \Longrightarrow$ Gives you a_3 in terms of a_0.

$n = 3 \Longrightarrow$ Gives you a_4 in terms of a_1, which is 0, so $a_4 = 0$.

$n = 4 \Longrightarrow$ Gives you a_5 in terms of a_2, which is 0, so $a_5 = 0$.

$n = 5 \Longrightarrow$ Gives you a_6 in terms of a_3, which goes back to a_0.

$\vdots$

a_0 is an arbitrary constant.

$a_1 = 0$ from above.

$a_2 = 0$ from above.

$$n = 2 \Longrightarrow a_3 = \frac{3a_0}{3} = a_0$$

$$n = 3 \Longrightarrow a_4 = \frac{3a_1}{4} = 0$$

$$n = 4 \Longrightarrow a_5 = \frac{3a_2}{5} = 0$$

$$a_6 = \frac{3a_3}{6} = \frac{a_0}{2}$$

$$a_9 = \frac{3a_6}{9} = \frac{a_0}{2 \cdot 3}$$

$$a_{3n} = \frac{a_0}{n!}$$

$$\begin{aligned} y &= a_0 + a_0 x^3 + \frac{a_0}{2} x^6 + \frac{a_0}{3!} x^9 + \cdots \\ &= a_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} \\ &= \boxed{ce^{x^3}} \end{aligned}$$

(Of course, this agrees with what you would have gotten solving it as a separable or linear DE.)

Example III

Guess a series solution to the differential equation:

$$y'' - xy' - y = 0$$

Plug in the solution and find a recurrence relation on the coefficients.

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n \\
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}, xy' = \sum_{n=1}^{\infty} n a_n x^n \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n
\end{aligned}$$

Strategy for combining series:

1. First match <u>exponents</u> on x by shifting indices using the mnemonic.
2. Then match <u>starting indices</u> by pulling out initial terms.

[We could have matched these at $n = 0$, but I kind of like this because it shows some starting terms.]

$$2a_2 - a_0 + \sum_{n=\boxed{1}}^{\infty} \left[(n+2)(n+1)a_{n+2} - na_n - a_n\right] x^n = 0$$

$$\begin{aligned}
2a_2 - a_0 &= 0 \implies a_2 = \frac{a_0}{2} \\
a_{n+2} &= \boxed{\frac{a_n}{n+2} \text{ for } n \geq \boxed{1}}
\end{aligned}$$

Example IV

Use the recurrence relation derived above to solve $y'' - xy' - y = 0$.

$$\begin{aligned} 2a_2 - a_0 &= 0 \\ \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - na_n - a_n \right] x^n &= 0 \end{aligned}$$

Example IV

Use the recurrence relation derived above to solve $y'' - xy' - y = 0$.

$$\begin{aligned} 2a_2 - a_0 &= 0 \\ \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - na_n - a_n \right] x^n &= 0 \end{aligned}$$

$$
\begin{aligned}
2a_2 - a_0 &= 0 \Longrightarrow a_2 = \frac{a_0}{2} \\
a_{n+2} &= \frac{a_n}{n+2} \text{ for } n \geq \boxed{1} \\
a_0, a_1 & \quad \text{are arbitrary.} \\
a_2 &= \frac{a_0}{2} \\
a_3 &= \frac{a_1}{3} \\
a_4 &= \frac{a_2}{4} = \frac{a_0}{2 \cdot 4} \\
a_5 &= \frac{a_3}{5} = \frac{a_1}{3 \cdot 5} \\
y &= a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_0}{2 \cdot 4}x^4 + \frac{a_1}{3 \cdot 5}x^5 + \cdots \\
&= a_0\left(1 + \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 + \cdots\right) + a_1\left(x + \frac{1}{3}x^3 + \frac{1}{3 \cdot 5}x^5 + \cdots\right) \\
&= a_0\left(1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2 \cdot 4 \cdot \cdots \cdot 2n}\right) + a_1\left(\sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)}\right) \quad \Big\{ \text{Multiply by } \frac{2 \cdot 4 \cdot 6 \cdots}{2 \cdot 4 \cdot 6 \cdots} \\
&= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + a_1 \sum_{n=0}^{\infty} \frac{2^n n! x^{2n+1}}{(2n+1)!} \\
&= \boxed{c_1 e^{\frac{x^2}{2}} + c_2 \sum_{n=0}^{\infty} \frac{2^n n! x^{2n+1}}{(2n+1)!}}
\end{aligned}
$$

One is an elementary function, and one is a power series. To evaluate it, plug in a value of x and take as many terms as you want for accuracy.

Example V

Guess a series solution to the differential equation:

$$y'' - 3xy' - 3y = 0$$

Plug in the solution and find a recurrence relation on the coefficients.

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n & 3y &= 3a_0 + \sum_{n=1}^{\infty} 3a_n x^n \\
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} & 3xy' &= \sum_{n=1}^{\infty} 3n a_n x^n \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} & &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \\
& & &= 2a_2 + \sum_{n=\boxed{1}}^{\infty} (n+2)(n+1) a_{n+2} x^n
\end{aligned}
$$

[We could have matched these at $n = 0$, but I kind of like this because it shows some starting terms.]

$$
\begin{aligned}
2a_2 - 3a_0 &= 0 \Longrightarrow a_2 = \frac{3a_0}{2} \\
(n+2)(n+1)a_{n+2} - 3na_n - 3a_n &= 0, \text{ for } n \geq \boxed{1} \\
\underline{\text{Recurrence:}} \quad a_{n+2} &= \boxed{\frac{3a_n}{n+2}}
\end{aligned}
$$

Example VI

Use the recurrence relation derived above to solve $y'' - 3xy' - 3y = 0$.

$$
\begin{aligned}
2a_2 - 3a_0 &= 0 \\
\underline{\text{Recurrence:}} \quad a_{n+2} &= \boxed{\frac{3a_n}{n+2}}
\end{aligned}
$$

Example VI

Use the recurrence relation derived above to solve $y'' - 3xy' - 3y = 0$.

$$
\underline{\text{Recurrence:}} \quad a_{n+2} = \boxed{\frac{3a_n}{n+2}}
$$

$$
\begin{aligned}
2a_2 - 3a_0 &= 0 \implies a_2 = \frac{3a_0}{2} \\
\underline{\text{Recurrence:}} \quad a_{n+2} &= \frac{3a_n}{n+2} \\
a_0, a_1 & \quad \text{are arbitrary.} \\
a_2 &= \frac{3a_0}{2} \text{ from above} \\
n = 1: \quad a_3 &= \frac{3a_1}{3} = a_1 \\
a_4 &= \frac{3a_2}{4} = \frac{3^2 a_0}{2 \cdot 4} \\
a_5 &= \frac{3a_3}{5} = \frac{3^2 a_1}{3 \cdot 5} \\
y &= a_0 + a_1 x + \frac{3a_0}{2}x^2 + \frac{3a_1}{3}x^3 + \frac{3^2 a_0}{2 \cdot 4}x^4 + \frac{3^2 a_1}{3 \cdot 5}x^5 + \cdots \\
&= a_0\left(1 + \frac{3}{2}x^2 + \frac{3^2}{2 \cdot 4}x^4 + \frac{3^3}{2 \cdot 4 \cdot 6}x^6 + \cdots\right) + a_1\left(x + \frac{3}{3}x^3 + \frac{3^2}{3 \cdot 5}x^5 + \frac{3^3}{3 \cdot 5 \cdots}\right. \\
&= a_0 \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}x^2\right)^n}{n!} + a_1 \sum_{n=0}^{\infty} \frac{3^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)} x^{2n+1} \qquad \left\{ \text{Multiply by } \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (} \right. \\
&= \boxed{c_1 e^{\frac{3}{2}x^2} + c_2 \sum_{n=0}^{\infty} \frac{6^n n!}{(2n+1)!} x^{2n+1}}
\end{aligned}
$$

Notes:

1. Sometimes (usually) you can't convert back to elementary functions.

2. Sometimes you can't even find a nice formula for the general term. In that case, just calculate the first few coefficients using the recursion relation.